

New multiplicativity results for qubit maps

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Abstract

Let Φ be a trace-preserving, positivity-preserving (but not necessarily completely positive) linear map on the algebra of complex 2×2 matrices, and let Ω be any finite-dimensional completely positive map. For $p = 2$ and $p \geq 4$, we prove that the maximal p -norm of the product map $\Phi \otimes \Omega$ is the product of the maximal p -norms of Φ and Ω . Restricting Φ to the class of completely positive maps, this settles the multiplicativity question for all qubit channels in the range of values $p \geq 4$.

1 Introduction and statement of results

Qubit maps provide a useful laboratory for exploring methods and conjectures in quantum information theory. In particular they can serve as a testing ground for approaches to the problem of additivity of minimal entropy, and the related issues of Holevo capacity and entanglement of formation [18]. In this paper we will focus on the maximal p -norm and consider the question of its multiplicativity for a product map, when one of the factors in the product is a qubit map. For values of p close to one this question is directly related to the additivity of minimal entropy, and hence to the circle of problems mentioned above.

Recall first that the Schatten norm of a matrix A is defined for $p \geq 1$ as

$$\|A\|_p = \left(\text{Tr} |A|^p \right)^{1/p} = \left(\text{Tr} (A^* A)^{p/2} \right)^{1/p} \quad (1)$$

Let Φ be a linear map on the matrix algebra $\mathbb{C}^{d \times d}$, then the maximal p -norm of Φ is defined as

$$\nu_p(\Phi) = \sup_{\rho} \|\Phi(\rho)\|_p = \sup_{|\psi\rangle} \|\Phi(|\psi\rangle\langle\psi|)\|_p \quad (2)$$

where the first sup runs over states in $\mathbb{C}^{d \times d}$, the second sup runs over pure states (normalized vectors in \mathbb{C}^d), and the second equality follows by convexity of the p -norm. It is natural to define another norm $\|\Phi\|_{1 \rightarrow p}$ by instead taking the sup over all matrices A satisfying $\|A\|_1 = 1$, and this has been considered in other work [19, 6]; however for the applications in this paper we are interested only in the quantity defined in (2). In the case $d = 2$ we will refer to Φ as a *qubit* map.

Recall that the map Φ is positivity-preserving if $\Phi(A) \geq 0$ for every $A \geq 0$, and trace-preserving if $\text{Tr } \Phi(A) = \text{Tr } (A)$. The map is completely positive (CP) if in addition $\Phi \otimes I_{d'}$ is positivity-preserving for every dimension d' . A *channel* is a CP, trace-preserving map.

Amosov and Holevo [2] conjectured that the maximal p -norm is multiplicative for products of channels, that is for any channels Φ and Ω and for all $p \geq 1$

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi) \nu_p(\Omega) \quad (3)$$

Later Holevo and Werner [20] found a family of d -dimensional channels Ψ for which $\nu_p(\Psi \otimes \Psi) > \nu_p(\Psi)^2$ for p sufficiently large ($p > 4.78 \dots$ for $d = 3$). No such example is known for $d = 2$, and the original conjecture (3) survives for the case where at least one of the channels Φ, Ω is a qubit channel.

In our main result we prove (3) for the case where Φ is a trace-preserving, positivity-preserving qubit map, where Ω is any finite-dimensional completely positive map, and where $p = 2$ or $p \geq 4$. We do *not* assume that Φ is completely positive. Indeed it is essential for our proof that we consider the larger class of positivity-preserving but not completely positive maps. Previous work on entrywise positive maps [14] has provided other examples where multiplicativity holds for a class of non-CP maps, in the range $p \geq 2$.

Theorem 1 *Let Φ be a trace-preserving, positivity-preserving qubit map, and Ω any finite-dimensional completely positive map. Then for $p = 2$ and for all $p \geq 4$,*

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi) \nu_p(\Omega) \quad (4)$$

There has been a lot of work on the additivity and multiplicativity question for quantum channels, and (4) has been established for special classes of qubit

channels, including the depolarizing channel [5, 3, 7, 1], unital qubit channels [15, 10], and some classes of non-unital qubit channels [11, 12, 17, 8]. Theorem 1 settles the question of multiplicativity for all qubit channels, at least in the range $p \geq 4$ (the case $p = 2$ was proved in [11]). It should be noted that (4) is false in general for positivity-preserving qubit maps if $p < 2$, as can be seen with the example $\Phi \otimes I$ where $\Phi(\rho) = \rho^T$. We are not aware of any counterexamples to (4) for $2 < p < 4$.

The proof of Theorem 1 uses the following matrix inequality which is of independent interest.

Theorem 2 *Let $A, B, C, D \in \mathbb{C}^{d \times d}$ for some $d \geq 1$. Then for $p = 2$ and for all $p \geq 4$,*

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_p \leq \left\| \begin{pmatrix} \|A\|_p & \|B\|_p \\ \|C\|_p & \|D\|_p \end{pmatrix} \right\|_p \quad (5)$$

The inequality (5) was first derived by M. Nathanson [16]. It had been known previously in the cases where $A = D$ and $B = C$ [4], where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is positive semidefinite [12], and where all matrices A, B, C, D are diagonal [13]. We conjecture that the inequality holds in the interval $2 \leq p \leq \infty$, and that the reverse inequality holds in the interval $1 \leq p \leq 2$ (it is easy to see that equality holds at $p = 2$). Proving this conjecture would also establish the non-commutative version of Hanner's inequality [4].

The paper is organised as follows. In Section 2 we prove Theorem 1 for a special sub-class of qubit maps, making use of the inequality (5). In Section 3 we recall a result of Gorini and Sudarshan [9] on the classification of extreme affine maps on \mathbb{R}^n which map the unit ball into itself. Combining the Gorini-Sudarshan classification with the representation of qubit maps as affine maps on \mathbb{R}^3 , we derive Lemma 4, which implies that any trace-preserving, positivity-preserving qubit map Φ can be expressed as a convex combination of qubit maps from the sub-class of Section 2, all of which share the same maximal output p -norm as Φ . Using Lemma 4, we then prove Theorem 1 for all qubit maps. Section 4 contains the proof of Theorem 2, which makes use of previously known matrix inequalities [12].

2 Proof for special class of maps

In this section we prove Theorem 1 for a special class of positivity-preserving, trace-preserving qubit maps. In order to describe this class we will use the

representation of qubit states by points in the Bloch sphere, and qubit maps by affine linear maps on \mathbb{R}^3 .

A qubit state ρ is represented by a point in the unit ball in \mathbb{R}^3 via the relation

$$\rho = \frac{1}{2}(I + \sum x_i \sigma_i) \mapsto x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (6)$$

where I is the identity matrix and $\{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices. Positivity of ρ is equivalent to

$$\sum x_i^2 \leq 1 \quad (7)$$

A trace-preserving qubit map Φ sends the state $\rho = \frac{1}{2}(I + \sum x_i \sigma_i)$ to the state $\Phi(\rho) = \frac{1}{2}(I + \sum y_i \sigma_i)$, where $y \in \mathbb{R}^3$ is obtained from x by applying an affine linear map, that is

$$y = Ax + v \quad (8)$$

for some real 3×3 matrix A , and some vector $v \in \mathbb{R}^3$.

Conjugation by a unitary matrix $U \in SU(2)$ maps ρ to $U\rho U^*$, and this acts on the Bloch sphere by a rotation, sending $x \mapsto R(U)x$ for some $R(U) \in SO(3)$. If unitary conjugations by matrices U, V are performed on the domain and range of the map Φ respectively, then the representation (8) is replaced by

$$y' = R(V)AR(U)x + R(V)v \quad (9)$$

Since the map $U \mapsto R(U)$ is onto, the singular value decomposition implies that it is always possible to find unitary matrices U, V so that $R(V)AR(U)$ is diagonal (though the diagonal entries need not be all positive). Spectral properties of the map Φ (in particular its maximal output p -norm) are invariant under unitary conjugations in its domain and range, hence there is no loss of generality in assuming that the matrix A in (8) is diagonal. Using the representation (8), we will say that Φ is in *diagonal form* if

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (10)$$

Note that Φ is unital if and only if $v = 0$ in (8). We now prove Theorem 1 for a special class of maps.

Lemma 3 *Let Φ be a positivity-preserving, trace-preserving qubit map in diagonal form (10), and suppose that at most one of the numbers (v_1, v_2, v_3) is nonzero. Then (4) holds for any completely positive map Ω , for $p = 2$ and for $p \geq 4$.*

Proof: By permuting coordinates we can assume that only the third component of v can be nonzero, so that $v_1 = v_2 = 0$. The diagonal entries of A may be positive or negative. However we can change the signs of any two diagonal entries by conjugating with a Pauli matrix, without destroying the diagonal property and without changing the third diagonal entry; for example conjugating with σ_3 changes the signs of λ_1 and λ_2 , and leaves λ_3 unchanged. Using this additional freedom we can assume that

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0 \quad (11)$$

Let ρ_{12} be a bipartite state on $\mathbb{C}^2 \otimes \mathbb{C}^d$ for some d , written in block form

$$\rho_{12} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \quad (12)$$

Let Ω be a completely positive map on \mathbb{C}^d , then

$$(I \otimes \Omega)(\rho_{12}) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad (13)$$

where $A = \Omega(X)$, $B = \Omega(Y)$ and $C = \Omega(Z)$. Since Ω is completely positive, and ρ_{12} is a state, it follows that $(I \otimes \Omega)(\rho_{12})$ is positive semidefinite, and hence $B = A^{1/2}RC^{1/2}$ where R is a contraction. This implies in particular that for all $p \geq 1$

$$\|B\|_p \leq \|A\|_p^{1/2} \|C\|_p^{1/2} \quad (14)$$

We will encounter the 2×2 matrices of p -norms

$$\begin{pmatrix} \|A\|_p & \|B\|_p \\ \|B\|_p & \|C\|_p \end{pmatrix}, \quad \begin{pmatrix} \|A\|_p & i\|B\|_p \\ -i\|B\|_p & \|C\|_p \end{pmatrix} \quad (15)$$

and we note now that (14) implies the positivity of these matrices, or more generally

$$\begin{pmatrix} \|A\|_p & z\|B\|_p \\ z^*\|B\|_p & \|C\|_p \end{pmatrix} \geq 0 \quad (16)$$

for any $z \in \mathbb{C}$ satisfying $|z| \leq 1$.

Using the diagonal form (10) and the assumption that v_3 is the only nonzero component of v , we have

$$(\Phi \otimes \Omega)(\rho_{12}) = \begin{pmatrix} c_{++}A + c_{+-}C & \lambda_1 B_1 - i\lambda_2 B_2 \\ \lambda_1 B_1 + i\lambda_2 B_2 & c_{--}A + c_{-+}C \end{pmatrix} \quad (17)$$

where $B = B_1 - iB_2$ with B_1, B_2 hermitian, and where

$$c_{+\pm} = (1 + v_3 \pm \lambda_3)/2, \quad c_{-\pm} = (1 - v_3 \pm \lambda_3)/2 \quad (18)$$

Since Φ is positivity-preserving, it maps the state $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ into a positive semidefinite matrix, and this implies that

$$c_{++} \geq 0, \quad c_{--} \geq 0 \quad (19)$$

Similarly it maps the state $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to a positive semidefinite matrix, hence also

$$c_{+-} \geq 0, \quad c_{-+} \geq 0 \quad (20)$$

2.1 The case $p = 2$

Using the representation (17),

$$\begin{aligned} \text{Tr} \left((\Phi \otimes \Omega)(\rho_{12}) \right)^2 &= \text{Tr} (c_{++}A + c_{+-}C)^2 \\ &+ 2(\lambda_1^2 \text{Tr} B_1^2 + \lambda_2^2 \text{Tr} B_2^2) + \text{Tr} (c_{--}A + c_{-+}C)^2 \end{aligned} \quad (21)$$

Using the positivity of the coefficients (19), (20) and convexity of the 2-norm gives

$$\begin{aligned} \text{Tr} \left((\Phi \otimes \Omega)(\rho_{12}) \right)^2 &\leq (c_{++}\|A\|_2 + c_{+-}\|C\|_2)^2 \\ &+ 2(\lambda_1^2 \text{Tr} B_1^2 + \lambda_2^2 \text{Tr} B_2^2) + (c_{--}\|A\|_2 + c_{-+}\|C\|_2)^2 \end{aligned} \quad (22)$$

Define

$$\lambda = \max\{\lambda_1, \lambda_2\} \quad (23)$$

then it follows that

$$\lambda_1^2 \text{Tr} B_1^2 + \lambda_2^2 \text{Tr} B_2^2 \leq \lambda^2 \text{Tr} B^* B = \lambda^2 \|B\|_2^2 \quad (24)$$

Using (24) the right side of (22) can be re-written as the trace squared of a 2×2 matrix, leading to

$$\begin{aligned} \text{Tr} \left((\Phi \otimes \Omega)(\rho_{12}) \right)^2 &\leq \text{Tr} \begin{pmatrix} c_{++}\|A\|_2 + c_{+-}\|C\|_2 & \lambda\|B\|_2 \\ \lambda\|B\|_2 & c_{--}\|A\|_2 + c_{-+}\|C\|_2 \end{pmatrix}^2 \\ &= \text{Tr} \left(\Phi \begin{pmatrix} \|A\|_2 & z\|B\|_2 \\ z^*\|B\|_2 & \|C\|_2 \end{pmatrix} \right)^2 \end{aligned} \quad (25)$$

where

$$z = \begin{cases} 1 & \text{if } \lambda = \lambda_1 \\ i & \text{if } \lambda = \lambda_2 \end{cases} \quad (26)$$

As noted in (16) the matrix $\begin{pmatrix} \|A\|_2 & z\|B\|_2 \\ z^*\|B\|_2 & \|C\|_2 \end{pmatrix}$ is positive semidefinite, hence by definition of the maximal 2-norm we get

$$\begin{aligned} \|(\Phi \otimes \Omega)(\rho_{12})\|_2 &\leq \left\| \Phi \begin{pmatrix} \|A\|_2 & z\|B\|_2 \\ z^*\|B\|_2 & \|C\|_2 \end{pmatrix} \right\|_2 \\ &\leq \nu_2(\Phi) \text{Tr} \begin{pmatrix} \|A\|_2 & z\|B\|_2 \\ z^*\|B\|_2 & \|C\|_2 \end{pmatrix} \\ &= \nu_2(\Phi) (\|A\|_2 + \|C\|_2) \end{aligned} \quad (27)$$

Since $A = \Omega(X)$ and $C = \Omega(Z)$, this yields

$$\begin{aligned} \|(\Phi \otimes \Omega)(\rho_{12})\|_2 &\leq \nu_2(\Phi) \nu_2(\Omega) (\text{Tr } X + \text{Tr } Z) \\ &= \nu_2(\Phi) \nu_2(\Omega) \end{aligned} \quad (28)$$

since $\text{Tr } X + \text{Tr } Z = \text{Tr } \rho_{12} = 1$. Since this holds for any state ρ_{12} we deduce that

$$\nu_2(\Phi \otimes \Omega) \leq \nu_2(\Phi) \nu_2(\Omega) \quad (29)$$

The inequality in the reverse direction follows by restriction to product states, hence this completes the proof for the case $p = 2$.

2.2 The case $p \geq 4$

We apply Theorem 2 to (17) to conclude that for $p \geq 4$,

$$\|(\Phi \otimes \Omega)(\rho_{12})\|_p \leq \left\| \begin{pmatrix} \|c_{++}A + c_{+-}C\|_p & \|\lambda_1 B_1 - i\lambda_2 B_2\|_p \\ \|\lambda_1 B_1 + i\lambda_2 B_2\|_p & \|c_{--}A + c_{-+}C\|_p \end{pmatrix} \right\|_p \quad (30)$$

Define the 2×2 real symmetric matrix

$$M = \begin{pmatrix} \|c_{++}A + c_{+-}C\|_p & \|\lambda_1 B_1 - i\lambda_2 B_2\|_p \\ \|\lambda_1 B_1 + i\lambda_2 B_2\|_p & \|c_{--}A + c_{-+}C\|_p \end{pmatrix}, \quad (31)$$

so that (30) can be written

$$\|(\Phi \otimes \Omega)(\rho_{12})\|_p \leq \|M\|_p \quad (32)$$

The positivity results (19) and (20) imply that

$$\begin{aligned} \|c_{++}A + c_{+-}C\|_p &\leq c_{++}\|A\|_p + c_{+-}\|C\|_p, \\ \|c_{--}A + c_{-+}C\|_p &\leq c_{--}\|A\|_p + c_{-+}\|C\|_p \end{aligned} \quad (33)$$

Furthermore, recall (23) and suppose first that $\lambda = \lambda_1$, so that $\lambda_1 - \lambda_2 \geq 0$. Then

$$\begin{aligned} \|\lambda_1 B_1 - i\lambda_2 B_2\|_p &= \|(\lambda_1 - \lambda_2)B_1 + \lambda_2 B\|_p \\ &\leq (\lambda_1 - \lambda_2)\|B_1\|_p + \lambda_2\|B\|_p \\ &\leq \lambda\|B\|_p \end{aligned} \quad (34)$$

where in the last inequality we used $\|B_1\|_p = \frac{1}{2}\|B + B^*\|_p \leq \|B\|_p$. A similar argument leads to the same conclusion if $\lambda = \lambda_2$.

We would like to replace the entries of M with the bounds on the right sides of (33) and (34), and argue that $\|M\|_p$ must increase under this substitution. However the matrix M may not be positive semidefinite (since Φ is not necessarily completely positive) so this is not immediately obvious. To see that it does in fact increase, let $p = 2q$ so that

$$\|M\|_p = \left(\|M^2\|_q\right)^{1/2} \quad (35)$$

Then the matrix $M^2 = M^*M$ is positive semidefinite with positive entries, and it is easy to see that this implies $\|M^2\|_q$ is an increasing function of the entries of M^2 . Since M is also entrywise positive, the entries of M^2 are increasing functions of the entries of M , and therefore so is $\|M^2\|_q$. Therefore $\|M\|_p$ increases when the bounds (33), (34) are inserted in the right side of (32), and we get

$$\|(\Phi \otimes \Omega)(\rho_{12})\|_p \leq \left\| \begin{pmatrix} c_{++}\|A\|_p + c_{+-}\|C\|_p & \lambda\|B\|_p \\ \lambda\|B\|_p & c_{--}\|A\|_p + c_{-+}\|C\|_p \end{pmatrix} \right\|_p \quad (36)$$

Now we note that the right side of (36) is unchanged if the upper-right entry $\lambda\|B\|_p$ is replaced by $z\lambda\|B\|_p$ and the lower left entry by $z^*\lambda\|B\|_p$ for any $|z| = 1$. Hence using the notation (26) again, (36) implies

$$\|(\Phi \otimes \Omega)(\rho_{12})\|_p \leq \left\| \Phi \begin{pmatrix} \|A\|_p & z\|B\|_p \\ z^*\|B\|_p & \|C\|_p \end{pmatrix} \right\|_p \quad (37)$$

We now repeat the arguments used above in the case $p = 2$, to conclude that

$$\begin{aligned} \|(\Phi \otimes \Omega)(\rho_{12})\|_p &\leq \nu_p(\Phi) (\|A\|_p + \|C\|_p) \\ &\leq \nu_p(\Phi) \nu_p(\Omega) (\text{Tr } X + \text{Tr } Z) \\ &= \nu_p(\Phi) \nu_p(\Omega) \end{aligned} \quad (38)$$

Since this holds for any state ρ_{12} we again deduce

$$\nu_p(\Phi \otimes \Omega) \leq \nu_p(\Phi) \nu_p(\Omega) \quad (39)$$

and this completes the proof for the case $p \geq 4$.

3 Reduction to special form

In this section we will show that the general case of Theorem 1 follows from Lemma 3. Recall that a trace-preserving, positivity-preserving qubit map Φ is represented by an affine linear map on \mathbb{R}^3 as in (8), sending the Bloch sphere (the closed unit ball in \mathbb{R}^3) into an ellipsoid. We will refer to the latter as the *image ellipsoid* of Φ .

For a positivity-preserving, trace-preserving qubit map Φ , the minimal output entropy and maximal output p -norm are all achieved on the same input state. That is, there is a pure state $|\psi\rangle$ such that for all $p \geq 1$

$$\nu_p(\Phi) = \sup_{\rho} \|\Phi(\rho)\|_p = \|\Phi(|\psi\rangle\langle\psi|)\|_p \quad (40)$$

Define the function

$$h_p(r) = \left(\left(\frac{1+r}{2} \right)^p + \left(\frac{1-r}{2} \right)^p \right)^{1/p} \quad (41)$$

The spectrum of $\Phi(|\psi\rangle\langle\psi|)$ is $\{(1 \pm r)/2\}$, for some $0 \leq r \leq 1$. Accordingly the value of (40) is

$$\nu_p(\Phi) = h_p(r) \quad (42)$$

We will denote by \mathcal{C}_r the set of all positivity-preserving, trace-preserving qubit maps whose maximal output p -norm is at most $h_p(r)$, that is

$$\mathcal{C}_r = \{\Phi : \nu_p(\Phi) \leq h_p(r)\} \quad (43)$$

Note that \mathcal{C}_r does not depend on p . Geometrically, \mathcal{C}_r consists of the positivity-preserving qubit maps for which the image ellipsoid lies inside the sphere of radius r centered at the origin.

It is clear that \mathcal{C}_r is a convex set. The next result shows that the extreme points of \mathcal{C}_r have a simple form. Recall the definition (10) of the diagonal form of a qubit map.

Lemma 4 *Let Φ be an extreme point in \mathcal{C}_r , represented in diagonal form by the affine map $x \mapsto Ax + v$ on \mathbb{R}^3 . Then at most one of the components of v is nonzero.*

Lemma 4 is a consequence of the following Theorem of Gorini and Sudarshan [9], which classifies all extreme affine maps of \mathbb{R}^n sending the closed unit ball into itself.

Theorem 5 [Gorini-Sudarshan] *Let D_n be the set of affine maps of \mathbb{R}^n which send the closed unit ball into itself. Denote by (B, w) the map $x \mapsto Bx + w$, where $w \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$. If (B, w) is an extreme point in D_n , then there are orthogonal matrices $Q_1, Q_2 \in O(n)$, and real numbers $0 \leq \kappa \leq 1$, $0 < \delta \leq 1$ such that*

$$Q_1 w = (0, \dots, 0, \delta(1 - \kappa^2)), \quad Q_1 B Q_2 = \text{Diag}(m, \dots, m, \kappa m) \quad (44)$$

where $m = \sqrt{1 + \kappa^2 \delta^2 - \delta^2}$ and where $\text{Diag}(d_1, d_2, \dots)$ denotes the diagonal matrix with entries d_1, d_2, \dots .

To derive Lemma 4 from Theorem 5, we identify \mathcal{C}_r with the set of scaled affine maps $rD_3 = \{(rB, rw) : (B, w) \in D_3\}$. Hence every extreme map Φ in \mathcal{C}_r corresponds to an affine map (rB, rw) where (B, w) satisfies (44). Furthermore the matrix Q_1 in (44) is in $O(3)$, and hence either $Q_1 \in SO(3)$ or $-Q_1 \in SO(3)$; similarly for Q_2 . Since every rotation in $SO(3)$ can be implemented by a unitary conjugation in $SU(2)$ (see the discussion leading up to (9)), this shows that Φ can be written in diagonal form with

$$A = \begin{pmatrix} \pm rm & 0 & 0 \\ 0 & \pm rm & 0 \\ 0 & 0 & \pm r\kappa m \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ \pm r\delta(1 - \kappa^2) \end{pmatrix}, \quad (45)$$

and this proves Lemma 4.

In the remainder of this section we will show that Theorem 1 follows from Lemma 3 and Lemma 4. Accordingly, suppose that Φ is a trace-preserving, positivity-preserving qubit map satisfying (42) for some $0 \leq r \leq 1$, so that

$$\nu_p(\Phi) = h_p(r) \quad (46)$$

Then it is sufficient to show that for any completely positive map Ω ,

$$\nu_p(\Phi \otimes \Omega) \leq h_p(r) \nu_p(\Omega) \quad (47)$$

Now \mathcal{C}_r is a closed bounded convex subset of \mathbb{R}^{12} (since the matrix A and vector v together have 12 entries), hence by Caratheodory's Theorem any element of \mathcal{C}_r can be written as a convex combination of at most 13 of its extreme points. The map Φ is in \mathcal{C}_r , hence there are extreme maps $\{\Phi_i\} \in \mathcal{C}_r$ such that

$$\Phi = \sum_i a_i \Phi_i \quad (48)$$

where $a_i \geq 0$ and $\sum a_i = 1$. Since $\{\Phi_i\}$ are in \mathcal{C}_r we also have

$$\nu_p(\Phi_i) \leq h_p(r) \quad (49)$$

Furthermore, combining Lemma 4 and Lemma 3, we deduce that

$$\nu_p(\Phi_i \otimes \Omega) = \nu_p(\Phi_i) \nu_p(\Omega) \leq h_p(r) \nu_p(\Omega) \quad (50)$$

for all i . By convexity of the p -norm it follows from (48) and (50) that

$$\begin{aligned} \nu_p(\Phi \otimes \Omega) &\leq \sum_i a_i \nu_p(\Phi_i \otimes \Omega) \\ &= \sum_i a_i \nu_p(\Phi_i) \nu_p(\Omega) \\ &\leq h_p(r) \nu_p(\Omega) \end{aligned} \quad (51)$$

and this proves (47).

4 Proof of Theorem 2

Let $p = 2q$ and define

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (52)$$

Then M^*M is positive semidefinite, and we write it in block form as

$$M^*M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad (53)$$

Now we apply the result of Theorem 1(b) from [12] to the matrix M^*M to deduce that

$$\|M^*M\|_q \leq \left\| \begin{pmatrix} \|M_{11}\|_q & \|M_{12}\|_q \\ \|M_{21}\|_q & \|M_{22}\|_q \end{pmatrix} \right\|_q \quad (54)$$

for all $q \geq 2$. Furthermore $M_{11} = A^*A + C^*C$, hence

$$\|M_{11}\|_q \leq \|A^*A\|_q + \|C^*C\|_q = \|A\|_p^2 + \|C\|_p^2 \quad (55)$$

Similarly

$$\|M_{12}\|_q = \|M_{21}\|_q \leq \|A\|_p \|B\|_p + \|C\|_p \|D\|_p \quad (56)$$

and

$$\|M_{22}\|_q \leq \|B\|_p^2 + \|D\|_p^2 \quad (57)$$

For a positive semidefinite 2×2 matrix the q -norm is an increasing function of the entries. Hence combining (54) with (55), (56), (57) gives

$$\|M^*M\|_q \leq \|m^*m\|_q \quad (58)$$

where

$$m = \begin{pmatrix} \|A\|_p & \|B\|_p \\ \|C\|_p & \|D\|_p \end{pmatrix} \quad (59)$$

Taking a square root of both sides gives

$$\|M\|_p \leq \|m\|_p \quad (60)$$

which is the stated result.

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